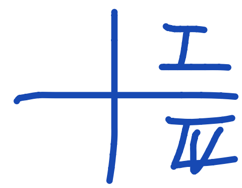
Inverse Trigonometric Functions
Module 3, Unit 7, Lesson 6

The Inverse Sine Function

The inverse sine function, denoted by $\sin^{-1}$, is the inverse of the restricted sine function $y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Thus,

$$y = \sin^{-1} x \quad \text{means} \quad \sin y = x$$



y = arc(sin x)

NOTE: $y = \sin^{-1} x$ **does not mean** $y = (\sin x)^{-1}$

$\downarrow$ Inverse sine function $\qquad$ $\downarrow$ Reciprocal of sine function

Finding Exact Values of $\sin^{-1} x$

1. Let $\theta = \sin^{-1} x$
2. Rewrite $\theta = \sin^{-1} x$ as $\sin \theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
3. Find the value of θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Example 1:

Find the exact value of

a. $\sin^{-1}\frac{\sqrt{2}}{2} = \boxed{\frac{\pi}{4}}$

b. $\sin^{-1}\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{3}}$

c. $\sin^{-1}\left(-\frac{1}{2}\right)$ $\boxed{-\frac{\pi}{6}}$

The Inverse Cosine Function

The inverse cosine function, denoted by $\cos^{-1}$, is the inverse of the restricted cosine function $y = \cos x$, $0 \le x \le \pi$. Thus,

$$y = \cos^{-1} x \quad \text{means} \quad \cos y = x$$

Finding Exact Values of $\cos^{-1} x$

1. Let θ = $\cos^{-1} x$
2. Rewrite θ = $\cos^{-1} x$ as cos θ = x, where $0 \le \theta \le \pi$
3. Find the value of θ in $[0, \pi]$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Example 2:

Find the exact value of

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$

b. $\cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$

c. $\cos^{-1} 1 = \boxed{0}$

The Inverse Tangent Function

The inverse tangent function, denoted by $\tan^{-1}$, is the inverse of the restricted tangent function y = tan x, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Thus,

$$y = \tan^{-1} x \quad \text{means} \quad \tan y = x$$

Finding Exact Values of $\tan^{-1} x$

1. Let θ = $\tan^{-1} x$
2. Rewrite θ = $\tan^{-1} x$ as tan θ = x, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3. Find the value of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan θ	und	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	0

Example 3:

Find the exact value of

a. $\tan^{-1}\sqrt{3}$ $\boxed{\frac{\pi}{3}}$

b. $\tan^{-1} 1$ $\boxed{\frac{\pi}{4}}$

c. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ $\boxed{-\frac{\pi}{6}}$

Inverse Properties

The Sine Function and Its Inverse

$\sin(\sin^{-1} x) = x$ for every x in the interval $[-1, 1]$

$\sin^{-1}(\sin x) = x$ for every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Cosine Function and Its Inverse

$\cos(\cos^{-1} x) = x$ for every x in the interval $[-1, 1]$

$\cos^{-1}(\cos x) = x$ for every x in the interval $[0, \pi]$

The Tangent Function and Its Inverse

$\tan(\tan^{-1} x) = x$ for every real number x

$\tan^{-1}(\tan x) = x$ for every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 4:

Find the exact value, if possible:

1. $\sin\left(\sin^{-1} 4.5\right) =$

$\boxed{\text{DNE}}$

2. $\sin^{-1}\left(\sin\frac{7\pi}{6}\right) =$

$\sin^{-1}\left(-\frac{1}{2}\right) =$ $\boxed{-\frac{\pi}{6}}$

3. $\cos^{-1}\left(\cos\frac{3\pi}{4}\right) =$

$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\boxed{\frac{3\pi}{4}}$

4. $\tan\left(\tan^{-1} 7\right) =$

$\boxed{7}$

5. $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) =$

$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) =$

$\boxed{-\frac{\pi}{6}}$

6. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) =$

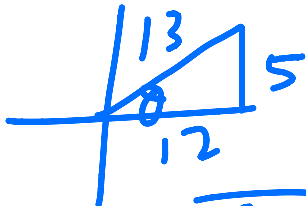
$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) =$

$\boxed{\frac{3\pi}{4}}$

Example 5:

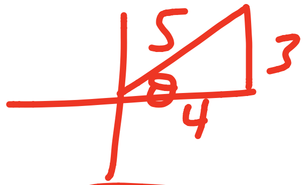
Find the exact value.

a. $\cos\left(\tan^{-1}\frac{5}{12}\right)$



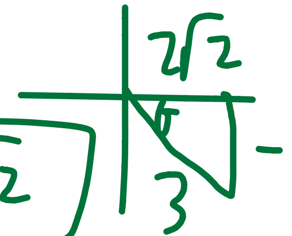
$\boxed{\frac{12}{13}}$

b. $\sin\left(\tan^{-1}\frac{3}{4}\right)$



$\boxed{\frac{3}{5}}$

c. $\cot\left[\sin^{-1}\left(-\frac{1}{3}\right)\right]$



$\boxed{-2\sqrt{2}}$

Practice

Find the exact value of each expression.

1. $\sin^{-1}1$

$\boxed{\frac{\pi}{2}}$

2. $\cos^{-1}1$

$\boxed{0}$

3. $\tan^{-1}1$

$\boxed{\frac{\pi}{4}}$

4. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\boxed{-\frac{\pi}{3}}$

5. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$\boxed{-\frac{\pi}{6}}$

6. $\sin(\cos^{-1}0)$

$\sin\left(\frac{\pi}{2}\right) =$

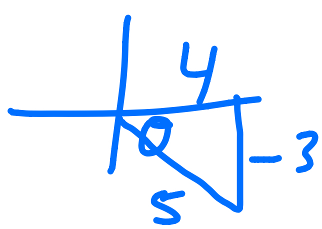
$\boxed{1}$

7. $\csc\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$

$\csc\left(\frac{\pi}{6}\right) =$

$\boxed{2}$

8. $\tan\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$



$\boxed{-\frac{3}{4}}$

9. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) =$

$\boxed{\frac{\pi}{3}}$